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PIERO DELLA FRANCESCA AND THE FOUNDATIONS OF LINEAR PERSPECTIVE

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Abstract: This article is a detailed study of the defense of linear perspective made by Piero della Francesca (c. 1474) in propositions 12th and 13th of Book 1st of *De prospectiva pingendi*. The purpose here is to elucidate the logic of argumentation, so as to sustain that it offers the foundation for the convergence of orthogonals in a perspectival representation, in opposition to Piero's statement that asserts that convergence is the foundation of the result shown in proposition 13th.

Keywords: Alberti, orthogonals, perspective, pictorial veil, observer.

PIERO DELLA FRANCESCA E OS FUNDAMENTOS DA PERSPECTIVA LINEAR

Resumo: Este artigo é um estudo detalhado da defesa da perspectiva linear feita por Piero della Francesca nas proposições 12ª e 13ª do Livro 1º do *De prospectiva pingendi*. O objetivo aqui é elucidar a lógica da argumentação, de maneira a sustentar que ela oferece o fundamento para a convergência de ortogonais em uma representação perspectiva, em oposição ao enunciado de Piero que afirma que a convergência é o fundamento do resultado mostrado na proposição 13ª.

Palavras-chave: Alberti, ortogonais, perspectiva, véu pictórico, observador.

1. Introduction

The theoretical developments, associated to the pictorial practices of the Renaissance in some Italian cities, undoubtedly contributed to change our relationship with space. Alberti's treatise *On painting* ([1435] 1966) is a milestone in such a theoretical endeavor. He is the first in raising to the category of theoretical awareness some successful practices in the art of painting from the workshops of young apprentices. In Alberti's treatise, practice acquires an awareness of method. Within the pictorial pressures of his time, not only religious interests are worth mentioning, but also the interest in developing detailed maps of large areas of the Earth,[1] which demanded the representation of three-dimensional realities on two-dimensional cardboard pieces. By then, the desire to represent depth on pictorial manifestations was not a novelty, the novelty was in the intention to achieve this with a geometrical key, following the degradation rhythm of lengths (linear perspective). Manetti, the first biographer of Brunelleschi, states the basic objective of the discipline, which during the Renaissance was given the name of *Perspectiva Artificialis*:

> [Perspective] forms part of that science which, in effect, consists of setting down properly and rationally the reductions and enlargements of near and distant objects as perceived by the eye of man: building, plains, mountains, places of every sort and location, with figures and objects in correct proportion to the distance in which they are shown (MANETTI, [1409] 1970, p. 42).

Alberti's basic proposal stems from the assumption that the painter must mainly devote attention to the way in which the visual pyramid, first conceived by Euclid as a conceptual instrument designed to give account of perceptual phenomena, is cut by a plane (pictorial veil) before its components converge upon the eye. In Alberti's words:

[1] By 1395, a sort of *study group* interested in Greek thought was formed in Florence. The members of the group were avid readers of Greek texts who did not hesitate to make long journeys only with the purpose of obtaining legendary works. In one of such travels they acquired a copy of Ptolemy's geography (EDGERTON, 1975, p. 93-94).

> He who looks a picture, done as I have described [above], will see a certain cross-section of a visual pyramid, artificially represented with lines and colours on a certain plane according to a given distance, centre and lights (ALBERTI, [1435] 1966, I, p. 52).

The square was the first object to attract the interest of perspective theorists. The concern for the square gradually defined a maxim that acquired the form of a methodological principle, which may be stated in the following words: "Give me an original square and its degradation on a pictorial panel as conceived for an observer in a clearly pre-established location, and I shall be able to represent any object laid upon the plane of the initial square for said observer". Alberti teaches how to degrade not a square, but a chessboard-like arrangement of squares; to this end, he first imagines the board laid upon the floor while the pictorial veil intersecting the visual pyramid is displayed perpendicular to the floor. Second, he takes into account the observer's location (distance regarding the pictorial plane and height) for whom the representation is conceived. The result obtained by the above described procedure on the pictorial veil may allow the creation of the chessboard illusion laid upon the floor for an observer located in the conceived position and height.

Alberti's exposition, written in a clearly humanistic tone, is not intended for mathematicians but rather for young painters wanting a certain approach to the pretended mathematical foundations of the pictorial practice. However, such foundations are not clearly shown. At the end of the First Book of the treatise, ALBERTI (1966, I, p. 58-59) states: "I usually explain these things to my friends with certain prolix demonstrations which in this commentary it seemed to me better to omit for the sake of brevity". Which are such demonstrations? Did they exist? Which friends had access to them? We do not know.

Piero della Francesca also wrote a treatise on painting.[2] Although it is written in Tuscan, perhaps because it is intended for a wide readership, it is

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[2] There is not a clear consensus among commentators about the date in which the treatise was conceived.

presented under a title in Latin, *De prospectiva pingendi*.[3] The Latin title maybe expresses the force with which the nature of science is claimed for the foundations of pictorial practice. By then, Piero did not use the current term *Perspectiva,* preferring a rather unusual term. It might be expected that with such decision the author wished to distinguish between the *Perspectiva Communis*,[4] on the one hand, a term coined to refer in general to the scientific study of visual perception, and, on the other hand, the *Perspectiva Artificialis*, a term coined during the Renaissance and referring to studies associated with pictorial techniques. Following this order of ideas we may then introduce Piero's treatise under the title of *Perspective at the service of pictorial arts*.

Piero's writing lacks the humanistic tone of Alberti's treatise. The former is conceived to provide technically the possible foundations for a pictorial practice closely related to Alberti's proposal. However, the exposition of such foundations exhibits many lacunae that should be filled by perspicacious readers. The presentation is tedious; although it is intended to be a deductive exposition, it does not clearly separate the assumed unquestionable starting point and the expected results. Otherwise, where the reader asks for a demonstration as a foundation, Piero gives instead an enumeration of steps, leading to a successful result. This presentation scheme is repeated over and over in different cases in which the same principles are applied. In instances where the exposition of a single case with its corresponding consistent demonstration is enough, Piero prefers to exhibit many cases, maybe in the hope that tedious repetition would engrave upon the reader's mind the

[3] There is also a version of the same written in Latin. Everything seems to indicate that Piero participated in the revision of such manuscript.

[4] Such was the name assigned to one of the works on perspective studies with the widest circulation in Europe. This is the work of John PECHAM ([1279] 1970), which basically gathers Roger Bacon's theoretical orientations and contributes to the diffusion of the most important works of Alhazen, the Arab philosopher. There are clear indications that such work was studied in strategic Italian centers, including Padova University, where Alberti was a student and the work was part of the official curriculum.

foundations of a technique. Such burdensome exposition style led Barbaro to qualify Piero's presentation as a writing for idiots (BARBARO, 1569, preface).

Notwithstanding the tediousness of Piero's presentation, propositions 12th and 13th in the First Book of his treatise are outstanding since they exhibit, in my opinion, what seems to be a clear geometric foundation of a successful geometric practice that allows adequate degradation on a square lying on the floor. The results of this practice match the method exposed by Alberti in his painting treatise.

The present paper is intended to make explicit the deep structure of Piero's argument; besides, I suggest the use of such structure as the foundation for the convergence of orthogonals assumed in Alberti's construction. However, I contest ELKINS (1987) attempt to introduce Piero's propositions as if they exhibited the assumed demonstration alluded by Alberti at the end of the First Book of this treatise. The scheme of the present paper is the following. First, I state the general problem, with a comment on the background given by Witelo's work. In the second part, I schematically introduce Alberti's solution; and third, I expose what is in my opinion the deep structure of Piero's argument, exhibited in propositions 12th and 13th. I also discuss the particular reading Elkins makes of the said propositions.

2. The problem

The problem we are going to focus upon goes as follows: given a square lying on the floor, a pictorial veil perpendicular to the floor, and an observer at the other side of the square in a clearly determined location in reference to the veil, one must find the shape that the square acquires when degradation on the pictorial veil occurs. The plane where the square is shall be called the *base plane*, while the intersection straight line between the base plane and the pictorial veil shall be named the *base line*. Due to pedagogical reasons that will not affect the generality of results, we will assume that one of the sides of the square lies on the base line. The precise determination of the location of the observer requires previous knowledge of the distance at which his feet are in reference to the base line (hereinafter *d*), the distance from his feet up to an elongation of one of the square's sides (hereinafter *m*), provided the veil is orthogonally faced, and, lastly, the height of the observer (hereinafter *h*). Therefore, whenever the observer's location is referred to, the triplet (*d, m, h*) should be clearly understood. The degraded form of the square corresponds to the figure that defines the edges of the intersection of the visual pyramid with the pictorial veil. The observer is the vertex of the pyramid while the square is its base.

One of the most accepted texts in Europe that summarized the most important breakthroughs in the studies of *Perspectiva communis*[5] was Witelo's *Perspectiva*, which proposed that when light beams from a point source passes through a square opening, the image formed on a wall obliquely placed to that opening acquires the shape of a trapeze. Witelo states:

> If the [luminous] ray [entering] through the middle of a square opening is obliquely incident to the surface of the opaque body laying underneath the surface of the aperture, the incident light will be of a shape [having] one part longer [than the other], [while] its angles will be equally bent. (WITELO, [1270] 1991, II, prop. 41)

A point source, lying in front of the medium point of a square opening, shall produce an equally square image upon a surface behind the opening, provided said surface is parallel to the opening. Otherwise, the image shall exhibit the shape of a quadrilateral with two opposite, unequal sides. Fig. 1 shows the result: a point source *A* located in the middle point of a square opening produces the image of a quadrilateral with unequal sides when projected upon an inclined wall lying behind that opening.[6] If instead of a light source, we imagine *A* is representing an eye, and instead of an opening we imagine that the

[5] Such breakthroughs included Euclid's developments in optics, Ptolemy's improvements in optics, Galen's anatomical research, Alhazen's contributions to optics and perception psychology, as well as the causal transmission mechanisms related to the visual pyramid studied by Roger Bacon in the Neoplatonic spirit of Robert Grosseteste.

[6] The geometrical models have been built with the Cabri II Plus and Cabri 3 D softwares.

initial square is a light source (either direct or reflected), and instead of a wall, we conceive a pictorial veil intersecting the visual pyramid between observer and square, there is not any difficulty assuming the image may acquire the shape of a trapeze.

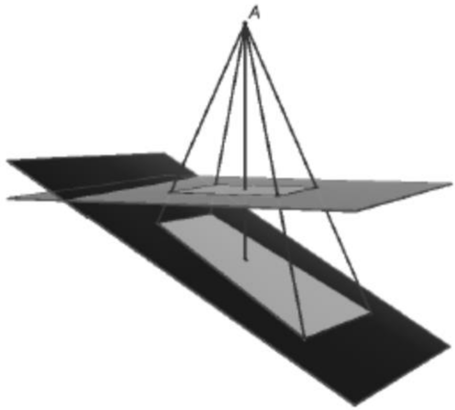

Figure 1. The shadow projected by a square opening forms a trapeze.

We can then reformulate the problem in the following terms: given a square lying upon the base plane, a pictorial veil perpendicular to said plane and in a location clearly determined, an observer on the other side of the square in reference to the veil, draw the trapeze obtained as a result of degrading the square on the pictorial veil. Finding the trapeze means determining the dimensions of its sides, its height, and its particular layout on the veil. Given, then, square *BCGF*, being *BC* the base line, the pictorial veil perpendicular to the base plane by the straight line *BC*, and an observer whose location is established by the triplet (d, m, h), determine the geometric characteristics of trapeze $BCG'F'$ (Fig. 2). Determination of the trapeze's characteristics demands finding the following: (i) the magnitude of segment $F'G'$, hereinafter x; (ii) the magnitude of height $F'F''$, hereinafter y; (iii) the magnitude of BF'', hereinafter z. In other words, finding the geometric properties of the trapeze means fully determining the triplet (x, y, z). Now then, such properties are not to be established from a mechanical device similar to the one in Fig. 1, instead they should be determined by methods we may name *a priori*, i.e., exclusively working with strokes upon a two-dimensional plane (on the pictorial veil).

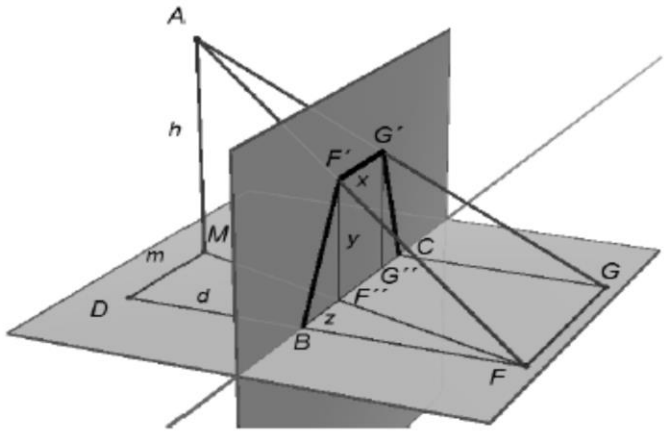

Figure 2. Formulation of the problem, given *A* observes the square *BCGF*, forming the trapeze *BCG´F´* on the pictorial veil.

3. Alberti's solution

Alberti introduces the solution to the problem in the last part of the First Book of his treatise. Such solution assumes two independent developments, each conceived on different planes, subsequently synthesized upon a drawing on the pictorial veil. Let's consider such developments separately.

First Development. Alberti requests one to draw, on a plane representing the pictorial veil, the *BC* base line. Above this line and at a height equivalent to the observer's height, he then requests one to fix point *I*, which he named *Central Point*. Upon this point converge degradations of straight lines arriving orthogonally to the base line. Alberti assumes this principle without providing any justification. Through this point, a parallel with the base line is drawn, later named *Horizon Line*, and one also draws a perpendicular line to the base line at point *B*. Segments *BL* and *LI* reproduce the observer's height and the distance from the observer's feet to the orthogonal arriving at *B*, i.e., *MA* and *DM* in Fig. 2. Locating point *I* in this way allows Alberti to bring a representation of the observer on the plane of the pictorial veil (Fig. 3).

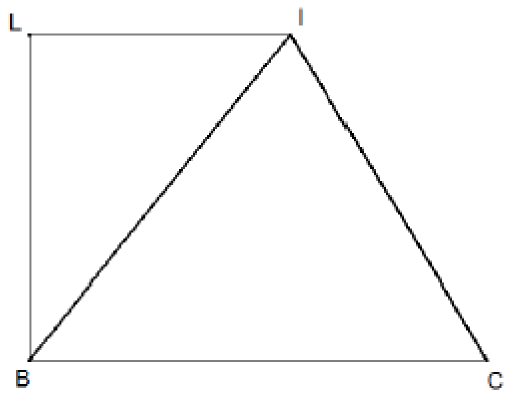

Figure 3. Convergence of orthogonals on the plane of the pictorial veil, with *I* at the height of the eye of the observer.

Second Development. Now, in order to determine how the lengths of the square segments, orthogonally extending towards the pictorial veil, diminish, Alberti requests the construction of a second drawing upon another sheet (Fig. 4). Now, he reproduces the length of the orthogonal *BF* segment, which matches the length of *BC*. Keeping the observer's height, *BL*, he now draws a parallel to *BF* by *L* and places point *A* upon it, in such a way that *AL* reproduces the distance from the observer's feet to the pictorial veil, i.e., *DB* in Fig. 2. He draws now the straight line *AF* and determines F_0 as the point of intersection with *BL*. The segment BF_0 shall define the height of the trapeze we are searching for.[7]

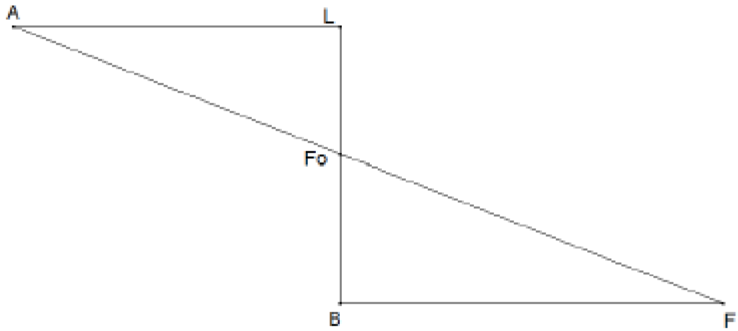

Figure 4. The evaluation of height of the trapeze, given by BF_0.

Synthesis. Now Alberti introduces segment BF_0 into the first diagram (Fig. 3), draws a parallel line to *BC* by F_0, and determines *F´* and *G´*, where the line respectively cuts the segments *IB* and *IC* (Fig. 5). This construction has already determined the elements we are searching for: *F´G´* has the adequate

[7] Cecil Grayson opened a discussion about *AL*'s location in reference to *BF*. It may be co-linear with one of the chessboard orthogonals he intends to degrade. In such a case, the observer would be located exactly in front of such orthogonal. Or it may be that *AL* is not co-linear anymore with *BF*, but parallel with the said orthogonal. In this case, the observer would be directly in front of an intermediate position between two chessboard orthogonals. The discussion is interesting for an accurate interpretation of the model Alberti had in mind. However, none of the two alternatives affects the general description I have made of Alberti's model (GRAYSON, 1964; GREEN & GREEN, 1987).

length *x*,[8] F_0B defines the trapeze's height *y*, and F_0F' determines the magnitude of *z*.

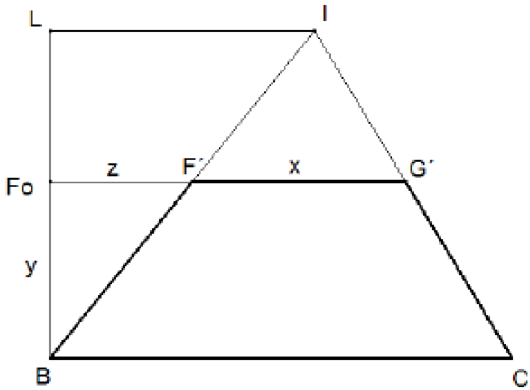

Figure 5. Synthesis.

The two developments may be gathered in the same figure (Fig. 6), always bearing in mind that below *BC* the figure depicts the base plane; above *BC* and to the right of *BL* the pictorial veil is depicted (first development); above *BC* and to the left of *BL* the construction advances upon a plane which is perpendicular both to the base plane and to the pictorial veil (second development). In this last case, it shall be assumed that given that *BC* and *BF* are congruent, point *C* now takes the place of point *F* (this is why it has been renamed *C-F*). Alberti uses without demonstrating or justifying, the following: (i) the convergence of orthogonals on any point located at the same height of the observer (*I*); (ii) the transverses upon the base plane (*FG*: parallel to *BC*) degrade in transverses on the pictorial veil (*F´G´*: parallel to *BC*).

[8] Nevertheless, Alberti does not offer any justification for the dimensions of *F´G´*; furthermore, he assumes that *I* is in an intermediate position, without indicating a way to extend the result to different positions.

Alberti proposes an argument that, in principle, should encourage the reader to consider that the method proposed is the correct one. ALBERTI (1966, I, p. 57) states: "If one straight line contains the diagonal of several quadrangles described in the picture, it is an indication to me whether they are drawn correctly or not". In other words, whatever the correct method is, it should preserve incidences; that is to say: if a point is in the intersection of two straight lines, such point's image is to be found on the cut of the images of the degraded straight lines; or, if a point is between other two points (on the same straight line), the image of said point shall be between the images of the other two (on the degradation of the original straight line). We shall call this the *Diagonal Criterium*.

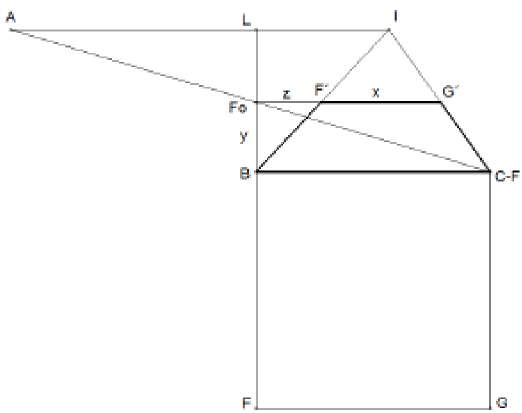

Figure 6. Alberti's proposal.

4. Deep structure in Piero's argument

Although, in his treatise, Piero della Francesca intends to offer the geometric foundation of pictorial techniques, the style of his demonstrations is not precisely adjusted to the style of an Euclidean demonstration. Piero not only fails to establish the axioms from which his results are derived, but he also fails to offer synthetic demonstrations as arguments. On the contrary, his argumentative lines are restricted to give only numerical examples. In most of the cases, Piero does not go beyond numerical examples, expecting the reader to attribute the power of a deep foundation to such cases. In his exposition, Piero assigns a name to each of the points he refers to, offering then descriptions of recommended procedures, in such a way that the reader may think that every description works as a sample which might be taken as a paradigmatic sample. In this order of ideas, each of the descriptions suggested in reference to particular points may act as a description referred to a variable or a set of variables. However, several reasons make the presentation rather ponderous for a contemporary reader. (i) The designation of names in a demonstration is not univocal, and Piero does not object granting the same name to two or three different points, thus making it difficult to identify them in the description offered in prose. (ii) In some instances, Piero uses numerals (1, 2, 3, ...) intermingled with letters as names assigned to points. (iii) In the absence of a functional language with elements that may operate as variables, Piero repeats in full detail the procedures to be followed in every case. If Piero had a functional scheme, he could have been able to describe properly the procedure to be followed by means of a set of points that could work as a paradigmatic sample, so as to suggest afterwards: "apply the same procedure with any point(s) that may be in a similar relation concerning the paradigmatic sample". This is not the case; thus, Piero describes and repeats as many similar procedures as applications are required. (iv) The intended demonstrations show, in many cases, lacunae that must be filled in by those readers willing to read Piero's work as if motivated by a rational construction (Principle of Charity).

Given the aforementioned difficulties, I intend to introduce Piero's contributions, which I consider substantial, but in a more friendly language intended for contemporary readers. I hope to be faithful to the spirit of the thinker. Piero's challenge is the same as assumed by Alberti, i.e.: to express a set of pictorial rules, together with their foundations, so that we may draw strokes upon a two-dimensional pictorial veil in reference to the spectator, whose position regarding the plane is known or clearly fixed beforehand, thus creating the illusion, at first-hand, of the contemplation of a sort of three-dimensional reality. Let's start quoting the definition of perspective given by the author in Book III:

> I say that perspective literally signifies things seen at a distance, represented as if they were enclosed within given limits [pictorial veil] and in a proportion adjusted to the magnitude of their distances. Knowledge, without which, nothing could be correctly degraded. And given the fact that painting is nothing without the demonstration of degraded surfaces or magnified upon the limit [pictorial veil], positioned as real things are seen by the eye, subtending different angles upon said limit, and considering that for any amount, some parts are closer to the eye than others, and the closest part always appears as if subtending a greater angle than the others on the assigned limit; and given the fact that it is not possible for the intellect to judge size by itself, this is, the size of the closest part, nor the rest, I state that [the use of] perspective is necessary, to distinguish proportionally all quantities, as a true science, demonstrating the degradation and magnification of all quantities by means of lines. (FRANCESCA, [1474] 1984, III, p. 128-9)

Apart from the kinship with definitions proposed by Alberti and Manetti, I would like to highlight the similarity of Piero della Francesca's opinions with the precepts recommended by Euclid, i.e., the dimensions and positions of an object are judged from an angle subtending the visual pyramid.

Considering the difficulties for modern readers when following Piero's exposition, many interpretations have been given to his proof: Martin KEMP (2000, p. 35), for example, believes that Piero's treatise may be seen as a logical extension of Albertian vision geometry; James ELKINS (1987) asserts that Piero's presentation is a copy of the proof that Alberti is supposed to have shared with his closest friends, and that Alberti refused to expose said proof in his treatise due to its mathematical complexity; Judith FIELD (2005, p. 134) states that there is no consistent evidence showing that Piero would have read Alberti's treatise prior to conceiving *De prospectiva*. As a matter of fact, Alberti writes in a humanistic style, wholly absent in Piero's treatise. I am going to defend a standpoint which is closer to Field's reading.

Elkins' argumentation presupposes the existence of the proof referred to by Alberti (without offering uncontroversial evidence), also presupposing that Piero could have been part of the circle of Alberti's closest friends (here, evidence is also insufficient).[9] Elkins states that propositions 12th and 13th are not harmonic with the rest of the treatise; this can be taken as a hint of the meddling of other line of argumentation. I see the following structure in Piero's First Book: (i) the first six propositions establish the acknowledgement of the basic principles stated in Euclid's optics; (ii) propositions 7th and 11th present some results (many of them elementary) about the invariance of proportions in the divisions practiced upon a segment when they are submitted to perspective transformations (these results are founded upon proportions stemming from the comparison of similar triangles); (iii) propositions 12th and 13th in which Piero unfolds his most original and astonishing results; (iv) the remaining propositions contain an important number of applications of previous results. From my viewpoint, in opposition to Elkins' suggestion, part (iii) displays continuity with part (i) since, as I demonstrate below, there are arguments to sustain that 12th and 13th come from Euclid's teachings on optics. Consequently, part (ii) is the one that seems to be in discontinuity with the remaining parts. Below, I suggest a presentation respecting the author's spirit but not the letter. There are some passages in which I try to fill the lacunae left by Piero's argumentation.

Let us consider again the problem exposed in Fig. 2, with the definitions of x, y, z, d, h, and m. It is clear that segment BC degrades in a segment identical to BC. Piero's construction is formed by three independent parts that afterwards, in a way not very clear in the exposition, are integrated into a single scheme. The order in which the three parts are developed is not

[9] ELKINS (1987, pp. 220, 222) quotes the following in his favor: (i) the fact that Piero lived for a while in Florence at the time Alberti's treatise was written; (ii) the fact that Alberti had acted in three instances as guarantor of Piero's works; (iii) the fact that Piero fulfilled the linguistic and mathematical requirements to attentively follow Alberti's arguments.

important at all. The first part establishes the dimensions the upper segment (*F´G´*, *x*) must have. I am going to identify the conclusion of this part under the term "Result 1". The second part (integrated to the first one in its development) determines the deformation of the trapeze (*BF´´, z*). I have named this conclusion "Result 2". The third part determines the height of the trapeze (*F´´F´, y*). The name I have given to this conclusion is "Result 3". The three parts are integrated to produce the final trapeze. Notwithstanding, Piero considers necessary an additional justification so as to provide a foundation that seems to be supported on "Result 4". I intend to demonstrate that Results 1, 2, and 3 cannot be explained or justified on the basis of Result 4 and that, on the contrary, the foundation of Result 4 can indeed stem from 1 and 3.

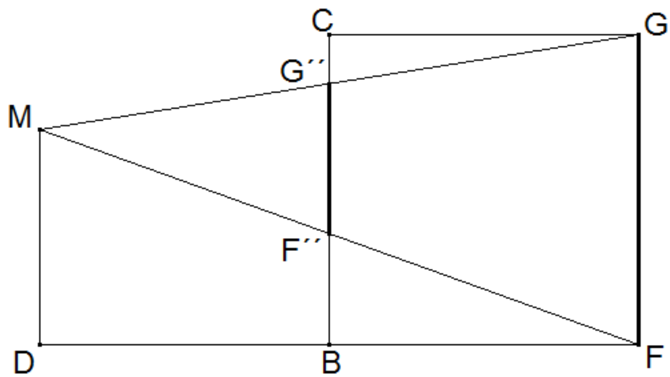

Figure 7. Degradation of GF.

Result 1. Let's imagine, now, that the height of the observer is reduced until it coincides with point *M*; that is, the observer is also on the base plane.[10]

[10] In this position all objects on the base plane are degraded in segments upon the base line.

It might be seen that, no matter the magnitude of segment *DM*, the length of *F´´G´´* is the same in all cases depending solely on the length of the square's sides and on the distance *DB*. The demonstration omitted by Piero is very simple and goes as follows (Fig. 7). Let *BC, CG, GF, FB* be the sides of a square, each of them with length *L*. Triangles *MGF* and *MG´´F´´* are similar, and therefore:

$$\frac{x}{L} = \frac{d}{L+d} \quad . \tag{1}$$

It can be inferred from the result that length *x* is independent of *m*. Although Piero does not say so, the theorem may be inversely read, that is: given a fixed segment *GF* and a straight *BC* parallel at a distance *L*, then all segments *F´´G´´* of length *x* laid upon *BC* may be laid in such a way that line segments *GG´´* and *FF´´* are found at equidistant points from *BC*. Piero adduces that from the viewpoint of *M*, *F´´G´´* is apparently the same size as *FG*, since the contemplation angle is the same. Piero accommodates himself, then, to Euclid's central principle, which suggests that the apparent size of contemplated objects depends on the angular width of the visual pyramid. This Euclidian predicament is introduced as the 2nd proposition in Book 1 of *De Prospectiva*. Following, I transcribe Piero's explanation adjusting the nomenclature to the one I am using:

> And the proportion from [*F´´G´´*] to [*FG*] is the same [given by] [*MF´´*] to [*MF*] [...][11] And when distances and things are in the same proportion than the eye's height and the thing degraded, it is clear then that degradation is correct. (FRANCESCA, [1474] 1984, I, Prop. 13th)

The last part of the quotation is obscure, but it might well be read in the direction expressed in Result 1.

Piero should also have to demonstrate that the length of degradation of *FG* (i.e., *F´G´*) is equally independent from the height of the observer (i.e.,

[11] It is clear that *MF´´* and *MF* are in the same proportion regarding *DB* and *DF*.

from *MA*, in Fig. 2). Piero does not offer any argumentation intended to defend such result. However, the result is easily demonstrable (Fig. 2).[12] From the similarity of triangles *AF´G´* and *AFG*, the following is inferred: $\frac{F'G'}{L} = \frac{AF'}{AF}$; from the similarity of triangles *AFM* and *F´FF´´*, the following is inferred: $\frac{AF'}{AF} = \frac{MF''}{MF}$; and from the similarity of triangles *MFD* and *F´´FB* the following is inferred: $\frac{MF''}{MF} = \frac{d}{L+d}$. Therefore, $\frac{F'G'}{L} = \frac{d}{L+d}$ and, thanks to (1), $F'G'=x$, no matter the magnitude of *MA*.

Piero's protocol allows, then, to find the dimensions of segment *F´G´* of the trapeze, that is, to represent the degraded square *BCGF* for an observer at distance *d* in reference to the pictorial veil, independently of his height and his position *DM*. This resolves the first component of the triplet sought for.

Result 3. Let's now try to establish the height of the trapeze.[13] If, without changing the height of the observer, we move *M* to make it coincide with *D*, point *F´´* shall then coincide with *B*. The line segment *AF* cuts the pictorial veil at point *Fo* in such a way that, according to Piero's conjecture, *BFo* determines the trapeze's height (Fig. 8). To defend his proposal, Piero refers to two arguments inspired on Euclid's optics. (i) If two parallel straight lines (in this case *BC* and *FG*) are perpendicular to a third one (*DF* in this case), and one draws from D a line, perpendicular to the plane of the other straight lines, establishing the position of an observer (*A* in this case) upon said perpendicular, then segment *FG* (the farthest) is perceived from point *A*, above the perception of *BC* (*De prospectiva,* 1, Prop. 6th) (EUCLID, 2000, prop. 10). (ii) Segment *BF*, seen from *A*, has the same appearance as *BFo*, because both are contemplated under the same angle (*De prospectiva*, 1st, prop. 2nd) (EUCLID, 2000,

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[12] I suppose, without any demonstration, that each of the points in segment *F´´F are* degraded in points upon segment *F´´F´* perpendicular to *BC.*

[13] First Piero presents what I have named *Result 3*, then he occupies himself with the so called *Result 1,* while *Result 2* is insinuated as something trivial. *Result 3 is* exposed in Proposition 12th.

def. 4). Given $BF_0 = y$, from the similarity of triangles *ADF* and F_0BF, the following is imposed:[14]

$$\frac{h}{y} = \frac{L+d}{L} \quad . \tag{2}$$

If transversals degrade in transversals (something Piero does not demonstrate), the height of the trapeze is independent of magnitude *m*. This solves the second component of the triplet sought for.

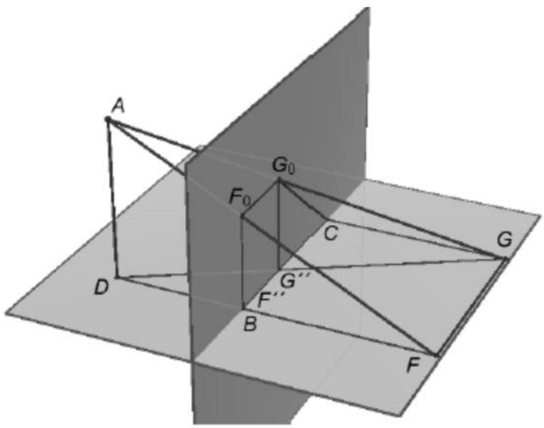

Figure 8. Height of the trapeze.

Result 4. Piero joins the two former constructions into a single one and suggests his particular method to degrade a square (Fig. 9). Below, I present a synthesis respecting the nomenclature I have proposed. The reader must pay

[14] This result fully matches with development 2nd of Alberti's argument.

attention to the change of representation occurring in each sheet of paper on which the drawing is made, since each case represents a different plane.

The paper seen as a perpendicular plane both to the base plane and to the pictorial veil. Piero assumes that the straight line *DBF* is an orthogonal to the pictorial veil, with $DB=d$, while the perpendicular segment *DA* represents the height of the observer, *h*. With the perpendicular by *B*, we find now the point where *AF* cuts the pictorial veil, which is F_0 (Result 3) the cut point. According to Piero's conjecture, BF_0 defines the height of the trapeze representing the degraded square.

The paper seen as the pictorial veil. Given the new function of the sheet of paper, we rename point *F* as point *C*, assuming *BC* as the base line. We draw now a parallel to *BF* by F_0, thus fixing the height where segment $F'G'$ must appear.

The paper seen as base plane. Let's imagine the square with its original dimension *BFGC* (*BC* is the base line).[15] Then, we draw straight lines *AF* and *AG* from *A* so as to establish the cuts with *BC*, i.e.: F'', G''. According to the theorem from equation (1), it is clear that the length of the segment $F''G''$ determines the length of the upper segment of the trapeze we are searching for. We already know that this result is independent of the magnitude of segment *DM* (Fig. 2); therefore, the evaluation can be carried out with a segment matching *DA*.

The paper seen anew as the pictorial veil. Upon the parallel by F_0 to the base line, which again makes it necessary to rename point *F* as point *C*,[16] we choose any point F' (with the characteristics I describe below) and we build segment $F'G'$ with the same length as $F''G''$.[17] The trapeze we are going to build above this line shows the pictorial representation on Alberti's veil. Then, it may be formulated as a conjecture that trapeze $BF'G'C$ is the presentation of a degraded square for an observer positioned at a distance *d* from the veil and

[15] Point *F* recovers its initial role.

[16] This exercise is easy since $BC \cong BF$.

[17] Here we radically give up Piero's order and exposition style due to reasons set out below.

whose height is *h*. Following Piero's synthesis is a complex task because one must pay attention to the change of the roles taken by segments *BC* and *BF-C*.

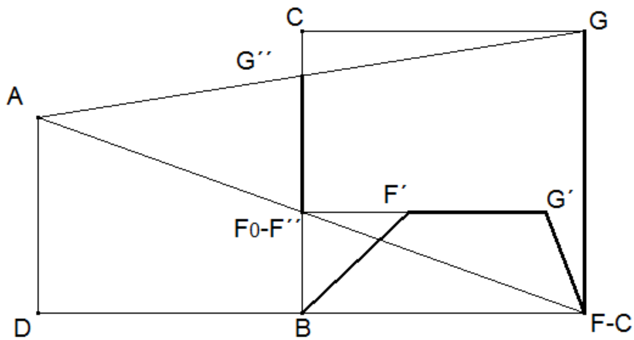

Figure 9. Piero's synthesis.

Following Piero, in order to demonstrate the conjecture, we must prove that the straight lines drawn from a point *I*, found at the same height as *A* and passing by points *B* and *C*, cut the parallel to *BF* that passes through F_0 in points *F´G´,* so that *F´G´* ≅ *F´´G´´*. Piero states it as follows, as if he were presenting the definite argument: "Proof: It is clear that [*F´G´*] is equal to [*F´´G´´*], which I have established as the appearance of the quantity [*FG*], as proved above"[18] (FRANCESCA, [1474] 1984, I, prop. 13th).[19] I shall name this "Result 4", which I summarize as follows: if the orthogonals meet at a point of the same height as *A*, then the length of *F´G´* matches the length of the trapeze representing a foreshortened square.

[18] Herein introduced as Result 1.

[19] I have adapted the nomenclature to the one I have been using during my exposition.

What I intend to defend is that the rational structure of Piero's proof does not require this argument as a foundation, since the justification is already complete when results 1 and 3 are put forward (such results are supported by Euclidian principles). In other words, Piero offers an unnecessary argumentation. Instead, he does not notice that the structure of the proof clearly allows the demonstration that all straight lines *BF´* and *CG´* (no matter the election made for point *F´*) cross at point *I*, located at the same height as the observer. Thus, the convergence of orthogonals proposed by Alberti is then well founded; it is also clear that both methods reach the same results. Let's name *w* the magnitude of the height of triangle *BIC* (Fig. 10). If we apply theorem (1) just the other way around, it becomes then clear that all triangles thus constructed satisfy the following equation:

$$\frac{x}{L} = \frac{w-y}{w} \quad . \tag{3}$$

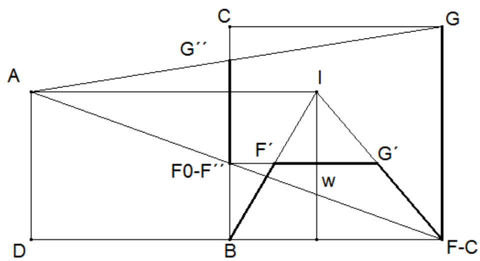

Figure 10. Convergence of orthogonals.

Therefore, when we vary the election of point *F´*, the vertex *I* of any triangle *BIC* travels along a straight line parallel to *BC* by *A*. Yet, it has to be proven that *w=h*. This result is simply drawn from eqs. (1), (2) and (3). From this result, it may be inferred as a corollary that all trapezes, representing the degradation of an original square from the viewpoint of a fixed observer, match in area, no matter what choice is made regarding point *F´*. In the presentation of proposition 13th, Piero concludes: "I shall say that [*F´G´.CG*] is the *BG´* plane degraded in square" (FRANCESCA, 1984, I, prop. XIII).[20]

This presentation of the argument differs from the argument given by Piero. I am not referring to my use of algebraic equations and expressions absent in Piero's exposition. The author reasons in the following terms (Fig. 11): draw a parallel to the base line by *A*, choose then upon it a point *I* (initially equidistant from *B* and from *F-C*), and draw the straight lines *IB* and *IC* to find the intersections *F´* and *G´* with the perpendicular to *BC* drawn from F_0. In order to prove that the trapeze built is the degraded image of the initial square, Piero shows that the length of segment *F´G´* matches the length of segment *F´´G´´*. At the end of the demonstration, Piero asks:

> Why do you place the eye in the intermediate position? [Piero refers to the choice of I]. Because I feel that it is the most convenient to appreciate the work; it can also be located wherever you want, provided that you do not go beyond the limits that are shown in the final figure [Book I],[21] and wherever you place it, you shall see everything in the same proportion (FRANCESCA, 1984, 1, Prop. 13th).

The difference between the literal approach of Piero's exposition and the spirit I perceive in the proof, stems from the fact that Piero admits (maybe as the working hypothesis) the convergence of the orthogonals, in order to evaluate the power of his method, while I consider that Piero's method already imposes the criterion of the perspective representation that would lead to a basis for the

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[20] This is an obscure text. The translation offered by Judith Field suggests a result that, in fact, would make Piero fall into a serious error. Field translates the passage in question as follows: "So I shall say that [the rectangle] *F´´G´´.CG* is the surface of [that is, it has the same area as] BF_0 made into a square", mentioning in a footnote that Piero is trying to say that $F''G''.CG = BF_0^2$. In the presentation she transcribes the notation I am using. There is no doubt that the result quoted by Field is wrong (FIELD, 2005, p. 364).

[21] Piero refers to a difficulty he glimpsed at the end of the Book 1st of *De Prospectiva,* that would lead him to establish some limits to his protocol of perspective projection.

convergence of orthogonals. Although Piero's demonstration develops as mentioned, I believe that deep down Piero's profound reasoning has to be adjusted to the model I have initially proposed. In this order of ideas, Piero's original reasoning may be adjusted to the following scheme. (i) First Premise of the form $q \supset p$: if the degraded orthogonals converge at a point located at the same height as the observer's, the magnitude of segment *F´G´* of trapeze *BF´G´C* matches the magnitude of segment *F´´G´´* (which has proven to be satisfactory for an observer at a pre-determined location, a condition imposed for the degradation of the farthest side of a square, provided that the trapeze's height is adjusted to the referred protocol). (ii) Second Premise *q*: segment *F´G´* adequately represents the degraded square we are interested in. From the form of the argument, we can only infer the possibility of *p*. If Piero's result is read as I have exposed, the first premise is $q \supset p$, the second then is *q*, and from them *p* might surely be inferred. That is to say: if segment *F´´G´´* matches the length of *F´G´*, then the orthogonals converge upon a point located at the same height as the observer's. Now then, since in fact by construction *F´´G´´* matches *F´G´* in length, we shall conclude the necessary convergence of orthogonals *BF´* and *CG´* at a point *I* at the same height as the observer.

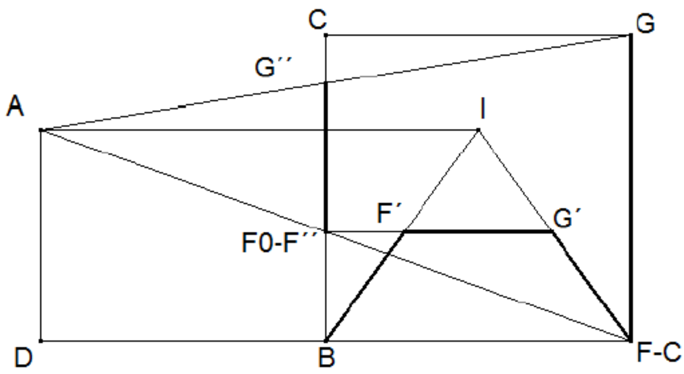

Figure 11. Piero's exposition.

The argument literally exposed by Piero has the following scheme: if we are sure of Result 4 as well as of Result 3, we may surely infer Result 1, which is the one granting the expected degradation. The scheme I propose is the following: since we are sure of Results 1 and 3, we can thus infer Result 4 (convergence of orthogonals). The reader remaining faithful to the letter must follow the first scheme, while the reader who prefers saving the order of reasons should choose the second scheme. While Piero has offered independent arguments (supported on Euclid's optics) for us to be sure of 1 and 3 without additional amendments, he has never mentioned his attachment to the principle of convergence of orthogonals. The fact that the author does not take the convergence of orthogonals as a central element, turning it into one of his assumptions, is an indication in favor of J.V. Field's thesis. Given the fact that the result $w=h$ does not depend at all on the length of the sides of the square, nor on the observer's distance, it is clear that all orthogonals converge towards I, not only those defining the orthogonal sides of the square, i.e., BF and CG.

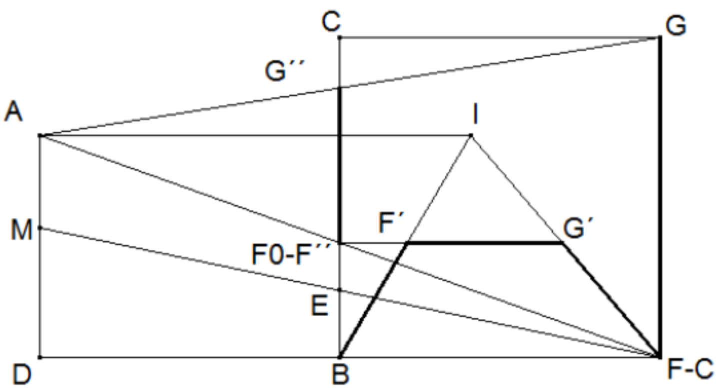

Figure 12. Choice of *F´*.

Result 2. Now, how must the choice of F' be established? Piero says nothing about it; there is no reason to do so, since his exposition is limited to the suggestion, particularly, that I is in an intermediate position. The painter assumes that extending the matter to other cases is a trivial matter. Though *De prospectiva* does not contain a solution, imagining it is not a complex endeavor. We can replicate the scheme of the synthesis and represent now point M on DA in such a way that DM shall match the distance from the observer's feet to the straight line DC (that is to say, z) (Fig. 12). In this case, the sheet of paper first represents the base plane, then the pictorial veil. We draw MF and determine E as the point of cut with BC. Lastly, we choose F' so that $F_0F' \cong BE$. Any point I, upon a parallel to BC passing by A, serves as a possible convergence of the orthogonals (as it is in fact supposed in Alberti's construction). The particular choice of I is determined by the distance from the observer's feet to the straight line DC. This resolves the third component of the triplet sought for.

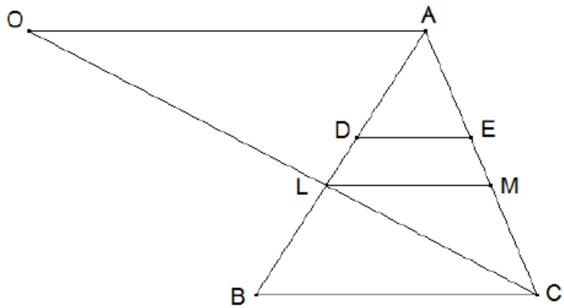

Figure 13. Proposition 23rd.

At the end of proposition 23rd of Book 1st, Piero suggests a result of far-reaching importance; though it may be easily inferred from previous results, he does not make any effort to demonstrate it. The author suggests another method to degrade a square, a method summoning all previous results. I quote Piero's presentation (Fig. 13):

> However if the size of the length [height] or of the width [of the trapeze] are unknown, I shall draw from point *A* a line parallel to *BC* corresponding in size [length] to the distance from the limit [pictorial veil] to that one given [position] for the eye and whose other extreme is point *O*, and from that line *OC* that shall divide line *BD* [a degraded orthogonal] at point *L*. I state that *BL* has taken in the degraded *BCDE* plane the quantity of *BC* […] If through L a parallel to *BC* is drawn cutting […] *CE* at point *M*, I state that *BLCM* is [represents] a square. (FRANCESCA, [1474] 1984, I, prop. 23rd).

Here, we have a brilliant result, introduced in an obscure prose. What Piero is trying to suggest is that the diagonals of a degraded square cut a straight line which is parallel to base line (*BC*), and that it passes by the convergence point of the orthogonals (*A*) at a point *O* located at a distance from *A,* equivalent to the distance between the observer and the pictorial veil. Let's see, then, the theorem from the general construction exposed in Fig. 14.

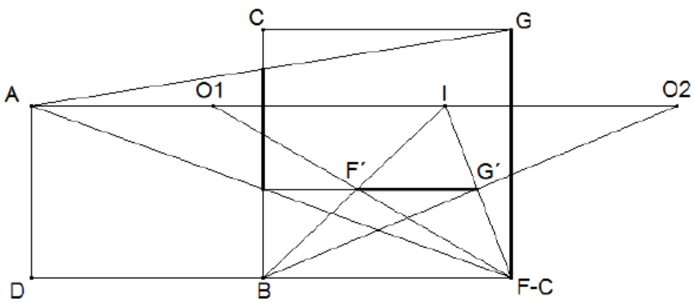

Figure 14. Convergence of orthogonals.

We shall prove that diagonals *CF´* and *BG´* cut the straight line *AI* (Horizon Line) at points O_1 and O_2, such that $O_1I = O_2I = d$. Triangles IO_1F' and $BF'C$ are similar, therefore: $\frac{O_1I}{L} = \frac{h-y}{y} = \frac{h}{y} - 1$; if we substitute Result (2) in said equation, we shall obtain: $\frac{O_1 I}{L} = \frac{d}{L'}$, which would lead us to what we wanted to demonstrate. The same applies if we had considered O_2. So then, the diagonal criterion is not an independent criterion to evaluate the goodness of the method, but rather a result which becomes a corollary for the assumed principles.

The protocols offered by Piero allow the resolution of a more general problem: given a point *P* on the base plane and an observer whose location regarding the pictorial veil is wholly known (*d, m, h*), find the *P* representation, i.e., *P´*, on the pictorial veil (Fig. 15). Although Piero does not formulate the general problem in any particular proposition, he does degrade many figures (pentagon, hexagon, octagon) which request degradation point to point.[22] No

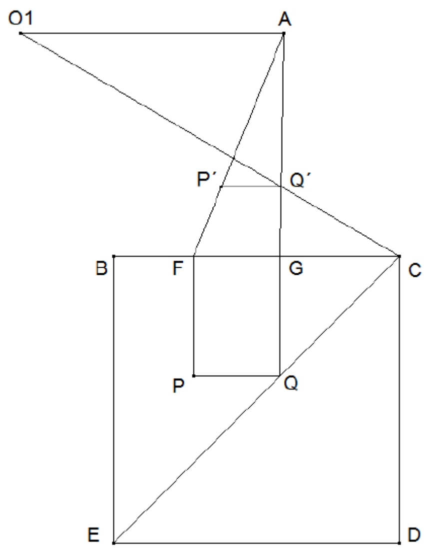

Figure 15. Representation of a point.

[22] This does happen in the following propositions of Book 1st: 16th, 17th, 18th, 19th, 20th.

universal rule is put forth, but the frequent (and boring) application in each particular case must lead the reader to draw a general rule. The rule is very simple; its foundation may be derived from the exercise of lying foundations for the degradation of a square and from the supposed conservation of incidences. From the description of the particular figures and the protocol to degrade a square, the following rule to degrade a point is drawn.

Being *P* a point on the base plane and *BCDE* a square within which is *P* (having *BC* on the base line), *A* is the convergence point of the orthogonals, therefore, it is located at a distance from *BC* matching the observer's height, O_1, upon the horizon line. This is one of the distance points (convergence of diagonals); consequently, the distance AO_1 matches the observer's distance to the pictorial veil. To find the image of *P*, consider the following procedure: (i) diagonal *EC* is drawn together with its degraded image CO_1; (ii) the parallel to *BC* is drawn by *P*, and cut *Q* is determined with the diagonal (*PQ* is a transversal); (iii) parallels to *BE* by *P* and by *Q* are drawn, and the cuts *F* and *G* are determined on the straight line *BC* (*PF* and *QG* are orthogonals); (iv) degraded orthogonals *FA* and *GA* are drawn; (v) the cut *Q´* of CO_1 and *GA* is determined (*Q´* is the image of *Q*); (vi) the parallel to *BC* by *Q´* is drawn, and the cut *P´* in reference to *FA* is determined (it is assumed, without any demonstration, that the images of transversals are transversals), with *P´* being the image of *P*.

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